Assignment 11.

Scwarz Lemma. Laurent Series.

This assignment is due Wednesday, April 15. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

1. Schwarz Lemma

Reminder. **Schwarz Lemma.** Let $f(z)$ be a function analytic on the disc $K : |z| < R$, $f(0) = 0$ and suppose that $|f(z)| \le M < \infty$ for all $z \in K$. Then $|f(z)| \le \frac{M}{R}|z|$ for all $z \in K$. Moreover, $|f'(0)| \le \frac{M}{R}$. Either equality is achieved if and only if f is a linear function $f(z) = e^{i\alpha}\frac{M}{R}z$, where $\alpha \in \mathbb{R}$.

(1) (a) Find a Möbius transformation g that sends the disc $K : |z| < R$ bijectively to itself, and sends 0 to $a \in K$.

(b) Generalize Schwarz Lemma to the case $f(a) = 0$ ($a \in \mathbb{C},\ |a| < R$). (*Hint:* Consider $f(g(z))$, where a Möbius transformation g is picked so that Scwarz Lemma applies to $f(g)$.)

2. Laurent Series

Notation. We sometimes use $\sum_{\mathbb{Z}}$ instead of $\sum_{n=-\infty}^{\infty}$.

Terminology. If for some function f it happens that $f(z) = \sum_{\mathbb{Z}} a_n(z - z_0)^n$ with $0 < |z - z_0| < R$, we say that $\sum_{\mathbb{Z}} a_n(z - z_0)^n$ is a Laurent expansion of f at z_0. If the $f(z) = \sum_{\mathbb{Z}} c_n z^n$ in an annulus $r < |z| < \infty$, we say that $\sum_{\mathbb{Z}} c_n z^n$ is a Laurent expansion of f at infinity.

(3) Suppose f has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

in an annulus $r < |z - z_0| < \infty$. Prove that $f(z)$ can be also represented in the form

$$f(z) = \sum_{n=-\infty}^{\infty} \tilde{a}_n z^n$$

in an annulus $\tilde{r} < |z| < \infty$ for some $\tilde{r}$.

Comment. This problem justifies use of the term "Laurent expansion at ∞" without specifying center z_0, since by the statement above we can choose z_0 to be 0.

(*Hint:* Use Laurent Series theorem.)

(4) Expand the function

$$f(z) = \frac{1}{(z - a)} \quad (a \in \mathbb{C},\ a \ne 0)$$

in a Laurent series in the annuli

(a) $0 < |z| < |a|$,

(b) $|a| < |z|$.

— see next page —

(5) Expand the function

$$f(z) = \frac{1}{(z-a)(z-b)} \quad (a, b \in \mathbb{C},\ 0 < |a| < |b|)$$

in a Laurent series in the annuli
(a) $0 < |z| < |a|$,
(b) $|a| < |z| < |b|$,
(c) $|b| < |z|$.
(*Hint:* Decompose into elementary fractions and use the previous problem.)

(6) Expand the function

$$f(z) = \frac{1}{(z-a)^k}, \quad (a \in \mathbb{C},\ a \neq 0, \quad k \in \mathbb{Z},\ k > 0)$$

in a Laurent series in the annuli
(a) $0 < |z| < |a|$,
(b) $|a| < |z|$.
(*Hint:* Use Weierstrass theorem).

(7) Expand each of the following functions in a Laurent series at the indicated points:
(a) $\frac{1}{z^2+1}$ at $z = i$ and $z = \infty$,
(b) $z^2 e^{1/z}$ at $z = 0$ and $z = \infty$.